ORDINARY DIFFERENTIAL EQUATIONS

Class: I M.Sc Maths
Subject: Ordinary Differential Equations
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Linear Equations with constant coefficients

Differential Equation:

A differential equation is an equation which contains derivatives of one or more depended variables with respect to one or more independent variables.

Example:

- $dy/dx = m$
- $\partial^2 y/\partial x^2 + \partial^2 y/\partial z^2 = 2xy$

Types of differential Equations:

Differential equations are classified as follows:

- Ordinary Differential Equation
- Partial Differential Equation

Ordinary Differential Equation:

A differential equation which involves only one independent variable.

Example:

$$dy/dx = m$$

Partial Differential Equation:

A differential equation which involves partial differential coefficient with respect to two or more independent variables

Example:

$$\partial^2 y/\partial x^2 + \partial^2 y/\partial z^2 = 2xy$$

Order and Degree of differential equation:

- The order of the highest order derivative occurred.
- The degree of the highest order derivative occurred.

Example:

$$(d^4y/dx^4)^2 + (dy/dx)^4 = e^x$$

- Order: 4
- Degree: 2

Linear differential equation:

A differential equation is said to be linear if the independent variable and its differential coefficients occur only in the first degree, and no product of dependent variable and/or is differential coefficient occur in the equation.

Non-Linear differential equation:

A differential equation which is not linear is called non- linear differential equation.

A linear differential equation of order n with constant coefficient is an equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots\ldots + a_n y = b(x),$$

where $a_0 \neq 0$, a_1, a_2, ….. , a_n are complex constants and b is complex valued function on an interval $I : a<x<b$.

The operator L defined by

$$L(\phi)(x) = \phi^{(n)}(x) + a_1\phi^{(n-1)}(x) + a_2\phi^{(n-2)}(x) + \ldots + a_n\phi(x)$$

is called as differential operator of order n with constant coefficients.

The equation $L(y) = b(x)$ is called non-homogenous equation. If $b(x) = 0$ for all x inI the corresponding equation $L(y) = 0$ is called a homogenous equation.

Initial Value Problems for Second Order Equations

We concerned with the equation $L(y) = y'' + a_1y' + a_2y = 0$ where a_1 and a_2 are constants.

Theorem:1

Let a_1, a_2 be constants and consider the equation $L(y) = y'' + a_1y' + a_2y = 0$

1. If r_1,r_2 are distinct roots of the characteristic polynomial $p(r) = r^2+a_1r+a_2$ Then the functions $\phi_1(x) = e^{r_1x}$ and $\phi_2(x) = e^{r_2x}$ are solutions of $L(y) = 0$.
2. If r_1 is a repeated root of the characteristic polynomial $p(r)$, the function $\phi_1(x) = e^{r_1x}$ and $\phi_2(x) = e^{r_2x}$ are solutions of $L(y) = 0$.

Proof:

Let $\phi(x) = e^{rx}$ be a solutions of $L(y) = 0$.

$$L(e^{rx}) = (e^{rx})'' + a_1(e^{rx})' + a_2(e^{rx})$$

$$= (r^2+a_1r+a_2)(e^{rx})$$

$L((e^{rx}) = 0$ iff $p(r) = r^2+a_1r+a_2 = 0$.

1. If r_1 and r_2 are distinct roots of p(r) then $L(e^{r_1 x}) = L(e^{r_2 x}) = 0$ and $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are solutions of L(y) = 0.

2. If r_1 is a repeated root of p(r), then

   $P(r) = (r - r_1)^2$ and $P'(r) = 2(r - r_1)$

   $L(e^{rx}) = P(r)e^{rx}$ for all r and x.

   $\partial/\partial r\, L(e^{rx}) = \partial/\partial r[P(r)e^{rx}]$

   which implies $L(xe^{rx}) = [P'(r) + xP(r)]e^{rx}$

   at $r = r_1$, $P(r_1) = P'(r_1) = 0$

   ie) $L(xe^{r_1 x}) = 0$ thus, showing that $x e^{r_1 x}$ is a solution of L(y) = 0.

Thus, if r_1 is a repeated root of the characteristic polynomial P(r), then $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are solutions of L(y) = 0.

THEOREM :2

If ϕ_1 and ϕ_2 are two solution of $L(y) = 0$ then $C_1 \phi_1 + C_2 \phi_2$ is also a solution of $L(y) = 0$, where C_1 and C_2 are any two constants.

PROOF:

Let ϕ_1 and ϕ_2 be two solutions of $L(y) = 0$

$$L(\phi_1) = \phi_1'' + a_1 \phi_1' + a_2 \phi_1 = 0$$

$$L(\phi_2) = \phi_2'' + a_1 \phi_2' + a_2 \phi_2 = 0$$

Suppose C_1 and C_2 are any two constants then the function ϕ defined by $\phi = C_1 \phi_1 + C_2 \phi_2$ is a solution of $L(y) = 0$

$$\begin{aligned} L(\phi) &= (a\, \phi_1 + C_2 \phi_2)'' + a_1(a\, \phi_1 + C_2 \phi_2) + a_2(a\, \phi_1 + C_2 \phi_2) \\ &= C_1(\phi_1'' + a_1 \phi_1' + a_2 \phi_1) + C_1(\phi_2'' + a_1 \phi_2' + a_2 \phi_2) \\ &= C_1 L(\phi_1) + C_2 L(\phi_2) \\ &= 0 \end{aligned}$$

The function ϕ which is zero for all x is also a solution called the trivial solution of $L(y) = 0$.

The results of above two theorems allow us to solve all homogeneous linear second Order differential equations with constant coefficients.

Definition:

An initial value problem $L(y) = 0$ is a problem of finding a solution ϕ satisfying $\phi(x_0) = \alpha_0$ and $\phi'(x_0) = \beta_0$ where x_0 is some real number α_0, β_0 are given constants.

Theorem: 3 (Existence Theorem)

For any real x_0 and constants α , β there exists a solution ϕ of the initial value problem $L(y) = y'' + a_1 y' + a_2 y = 0$, $y(x_0) = \alpha$, $y'(x_0)\ \beta$, $-\infty < x < \infty$.

Proof:

By theorem 1, there exist two solutions ϕ_1 and ϕ_2 that satisfy $L(\phi_1) = L(\phi_2) = 0$. From theorem 2, we know that $C_1\,\phi_1 + C_2\,\phi_2$ is a solution of $L(y) = 0$. We show that there are unique constants C_1, C_2 such that $\phi = C_1\,\phi_1 + C_2\,\phi_2$ satisfies $\phi(x_0) = \alpha$ and $\phi'(x_0) = \beta$.

$\phi(x_0) = C_1\phi_1(x_0) + C_2\phi_2(x_0) = \alpha$

$\phi'(x_0) = C_1\phi'_1(x_0) + C_2\phi'_2(x_0) = \beta$

Above system of equations will have a unique solution C_1, C_2 if the determinant

$$\Delta = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi'_1(x_0) & \phi'_2(x_0) \end{vmatrix}$$

$$= \phi_1(x_0)\,\phi'_2(x_0) - \phi_2(x_0)\,\phi'_1(x_0) \neq 0$$

By theorem 1, (1) $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are two solution of $L(y) = 0$ for $r_1 \neq r_2$ and

$$\Delta = e^{r_1 x_0}\, r_2\, e^{r_2 x_0} - e^{r_2 x_0}\, r_1\, e^{r_1 x_0}$$

$$= (r_2 - r_1)\, e^{(r_1 + r_2) x_0} \neq 0$$

By theorem1,(2) $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are two solution of $L(y) = 0$ and

$$\Delta = e^{r_1 x_0} [e^{r_1 x_0} + x_0 r_1 e^{r_1 x_0}] - x_0 e^{r_1 x_0} r_1 e^{r_1 x_0}$$
$$= e^{2r_1 x_0} \neq 0$$

Thus , the determinant condition is satisfied in both the cases. Therefore C_1, C_2 are uniquely determined. The function $\phi = C_1 \phi_1 + C_2 \phi_2$ is desired solution of the initial value problem.


thank you